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An Analytical Solution to Diffraction Problems Involving Periodically Structered Objects

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Abstract

A rigorous analysis of diffraction by rectangularly profiled, dielectric gratings is presented. An analytical solution is obtained for the total electromagnetic field. The approach presented can readily be extended to computations for metallic gratings and arbitrarily shaped surface-relief gratings.

1 Introduction

Because of their many potential applications, two-dimensional (2-D) dielectric gratings are of great interest. In these applications the gratings typically assume functions of beam combiners, splitters, or shapers, etc. [1], [2].

The design of such gratings inherently is an inverse problem, in our case involving Maxwell's equations with boundary conditions. Genetic Algorithms (GA) have shown promise of coping with this kind of problem. In [3], an excellent overview is provided on the application of GA to engineering electromagnetics.

Genetic Algorithms are stochastic search and optimization techniques with many advantages. They tend to produce globally optimal results, and the target functions are not required to be continuous. Another remarkable property is the inherent parallelism of GA so that they are ideally suited for implementation on a parallel machine.

For a CAD tool to be useful in the design of 2-D gratings based on GA, an efficient method for analysis is needed. With further extensions, the method presented here has the required properties. Modified slightly, it has been succesfully applied to analyze Multi-Quantum-Well DFB laser diodes in a three-dimensional manner as demonstrated in [4].

In [5], a system of second-order differential equations has been introduced to describe the total electromagnetic field within a rectangularly profiled, dielectric grating[1]. This system can be regarded as a wave equation with propagation constant. In this paper, an analytical solution to this wave equation is presented. For the sake of convenience, the wave equation will be briefly derived.

[1]... also called binary dielectric grating.

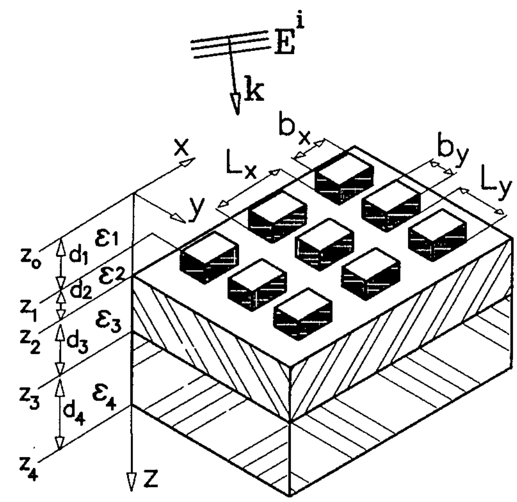

Figure 1: A binary, biperiodic grating exposed to the field of an incident plane wave.

2 Theory

In the scattering of a *homogeneous plane* wave $\mathbf{E}^i(\mathbf{r}) = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$, characterized by its polarization vector $\mathbf{E}_o$ and its wave number vector $\mathbf{k}$, by a biperiodically structured object, as illustrated in Fig.1, the total electromagnetic field obeys the Floquet theorem, i.e.

$$[\mathbf{E}(\mathbf{r}-\mathbf{r}_{mn}), \mathbf{H}(\mathbf{r}-\mathbf{r}_{mn})] = e^{j\mathbf{k}\cdot\mathbf{r}_{mn}}[\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r})] \quad (1)$$

for any $\mathbf{r}$, where $\mathbf{r}_{mn} = mL_x\mathbf{u}_x + nL_y\mathbf{u}_y$. Hence the total field can be expressed by

$$\begin{bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{bmatrix} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \begin{bmatrix} \mathbf{E}_{mn}(z) \\ \mathbf{H}_{mn}(z) \end{bmatrix} e^{-j\alpha_m x} e^{-j\beta_n y} \quad (2)$$

with $\alpha_m = \mathbf{k}\cdot\mathbf{u}_x + m2\pi/L_x$ and $\beta_n = \mathbf{k}\cdot\mathbf{u}_y + n2\pi/L_y$. L_x and L_y denote the periodicity in $\mathbf{u}_x$ and $\mathbf{u}_y$ direction, respectively. The vector expansion coefficients $\mathbf{E}_{mn}(z)$ and $\mathbf{H}_{mn}(z)$ are the unknowns which must be solved.

Slicing a scatterer in layers, the permittivity $\varepsilon(x, y, z)$ of each layer at constant z can be expanded into a 2-D Fourier series

$$\varepsilon(x, y, z) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \tilde{\varepsilon}_{mn}(z) e^{-jm\frac{2\pi}{L_x}x} e^{-jn\frac{2\pi}{L_y}y}. \quad (3)$$

In a homogeneous layer, $\varepsilon(x, y, z)$ simply equals $\bar{\varepsilon}_{00}$. Note that in Eqs. (2) and (3), M and N approach infinity.

Substitution of expansions (2) and (3) into Maxwell's equations yields after some manipulations

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{U}(z) + \omega_o^2 \mathbf{LC}\,\mathbf{U}(z) \tag{4}$$

$$\mathbf{I}(z) = (-j\omega_o \mathbf{L})^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathbf{U}(z) \tag{5}$$

where the following vectors

$$\mathbf{U}^T = [\mathbf{u}_x^T[\ldots, \mathbf{E}_{\nu(m,n)}, \ldots],\ \mathbf{u}_y^T[\ldots, \mathbf{E}_{\nu(m,n)}, \ldots]] \tag{6}$$

$$\mathbf{I}^T = [\mathbf{u}_x^T[\ldots, \mathbf{H}_{\nu(m,n)}, \ldots], -\mathbf{u}_y^T[\ldots, \mathbf{H}_{\nu(m,n)}, \ldots]] \tag{7}$$

with a bijective index transformation $\nu = (2N+1)(m+M)+n+N+1$ are defined. Further, denoting the Kronecker product with the symbol $\otimes$, the identity matrix by $\mathbf{1}$, and letting $k_o = \omega_o\sqrt{\mu_o\varepsilon_o}$, diagonal matrices

$$[\bar{\alpha}] = \frac{1}{k_o}[(\mathbf{k}\cdot\mathbf{u}_x)\mathbf{1} + \frac{2\pi}{L_x}\mathrm{diag}([-M, \ldots, M]\otimes \overbrace{[1, \ldots, 1]}^{2N+1})] \tag{8}$$

$$[\bar{\beta}] = \frac{1}{k_o}[(\mathbf{k}\cdot\mathbf{u}_y)\mathbf{1} + \frac{2\pi}{L_y}\mathrm{diag}(\underbrace{[1, \ldots, 1]}_{2M+1}\otimes[-N, \ldots, N])] \tag{9}$$

and the block matrices become

$$\mathbf{L} = \mu_o \begin{bmatrix} \mathbf{1} - [\bar{\alpha}]\mathcal{N}^{-2}[\bar{\alpha}] & -[\bar{\alpha}]\mathcal{N}^{-2}[\bar{\beta}] \\ -[\bar{\beta}]\mathcal{N}^{-2}[\bar{\alpha}] & \mathbf{1} - [\bar{\beta}]\mathcal{N}^{-2}[\bar{\beta}] \end{bmatrix} \tag{10}$$

$$\mathbf{C} = \varepsilon_o \begin{bmatrix} \mathcal{N}^2 - [\bar{\beta}]^2 & [\bar{\beta}][\bar{\alpha}] \\ [\bar{\alpha}][\bar{\beta}] & \mathcal{N}^2 - [\bar{\alpha}]^2 \end{bmatrix} \tag{11}$$

where the block Toeplitz-matrix

$$\mathcal{N}^2 = \begin{bmatrix} & & \vdots & & \\ & \bar{\mathbf{e}}_0 & \bar{\mathbf{e}}_{-1} & \bar{\mathbf{e}}_{-2} & \\ \cdots & \bar{\mathbf{e}}_{+1} & \bar{\mathbf{e}}_0 & \bar{\mathbf{e}}_{-1} & \cdots \\ & \bar{\mathbf{e}}_{+2} & \bar{\mathbf{e}}_{+1} & \bar{\mathbf{e}}_0 & \\ & & \vdots & & \end{bmatrix} \tag{12}$$

with

$$\bar{\mathbf{e}}_m = \begin{bmatrix} & & \vdots & & \\ & \bar{\varepsilon}_{m,0} & \bar{\varepsilon}_{m,-1} & \bar{\varepsilon}_{m,-2} & \\ \cdots & \bar{\varepsilon}_{m,+1} & \bar{\varepsilon}_{m,0} & \bar{\varepsilon}_{m,-1} & \cdots \\ & \bar{\varepsilon}_{m,+2} & \bar{\varepsilon}_{m,+1} & \bar{\varepsilon}_{m,0} & \\ & & \vdots & & \end{bmatrix} \tag{13}$$

is introduced in Eqs. (4) and (5).

Evidently, the diffraction of electromagnetic waves by objects with periodic structure can be modelled by coupled transmission lines [6], because they are also described by a wave equation of exactly the same form as given in Eqs. (4) and (5). It has to be pointed out that the definition of the matrix for the refractive index squared, $\mathcal{N}^2$, allows this compact formulation.

As in transmission line theory, the square of a propagation constant in z-direction can be defined in Eq. (4):

$$[\gamma_z]^2 = -\omega_o^2 \mathbf{LC} \tag{14}$$

Accordingly, a similar wave equation with propagation constant $-\omega_o^2\mathbf{CL}$ can also be deduced for the magnetic field.

Note, that the presented formulation is *exact*, because expansions (2) and (3) for $M, N \to \infty$ are both *rigorous*.

3 Analytical Solution

The system of second-order differential equations (4) can easily be solved by diagonalizing $[\gamma_z]^2$. If $\mathbf{P}$ contains the right eigenvectors of $[\gamma_z]^2$, the propagation constants of the waves in the decoupled domain are given by the following diagonal matrix

$$[\tilde{\gamma}_z] = \pm\sqrt{\mathbf{P}^{-1}[\gamma_z]^2\mathbf{P}} \tag{15}$$

A characteristic admittance and impedance can be defined as is also done in transmission line theory:

$$\mathbf{Y}_o = \mathbf{Z}_o^{-1} = \mathbf{P}^{-1}(j\omega_o\mathbf{L})^{-1}\mathbf{P}\,[\tilde{\gamma}_z] \tag{16}$$

$$= \mathbf{P}^{-1}(j\omega_o\mathbf{C})\mathbf{P}\,[\tilde{\gamma}_z]^{-1} \tag{17}$$

The solutions to the wave equation (4) are exponential functions. After rearranging terms, the analytical solutions can assume the form

$$\begin{bmatrix} \mathbf{U}(z) \\ \mathbf{I}(z) \end{bmatrix} = \underbrace{\mathbf{T}\tilde{\mathbf{A}}(z)\mathbf{T}^{-1}}_{\mathbf{A}(z):=} \begin{bmatrix} \mathbf{U}(z_0) \\ \mathbf{I}(z_0) \end{bmatrix} \tag{18}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}\mathbf{Y}_o \end{bmatrix} \tag{19}$$

and

$$\tilde{\mathbf{A}}(z) = \begin{bmatrix} \cosh([\tilde{\gamma}_z]z) & \sinh([\tilde{\gamma}_z]z) \\ \sinh([\tilde{\gamma}_z]z) & \cosh([\tilde{\gamma}_z]z) \end{bmatrix} \tag{20}$$

Matrix $\mathbf{A}$ denotes the chain matrix for one layer. For a homogeneous layer, the eigenvalues are explicitly given so that the chain matrix simplifies accordingly.

The electromagnetic field in a structure consisting of cascaded layers is easily obtained once the chain matrix of each layer has been computed. If $\mathbf{A}_i$ corresponds to the chain matrix of layer i in Fig. 1, the electromagnetic field in a plane z, $z_3 \le z \le z_4$ is then given by the product of the respective chain matrices:

$$\begin{bmatrix} \mathbf{U}(z) \\ \mathbf{I}(z) \end{bmatrix} = \mathbf{A}_4(z - z_3)\mathbf{A}_3(d_3)\mathbf{A}_2(d_2)\mathbf{A}_1(d_1)\begin{bmatrix} \mathbf{U}(z_0) \\ \mathbf{I}(z_0) \end{bmatrix} \tag{21}$$

In the same manner, the total electromagnetic field in arbitrarily biperiodic structures can be computed to any desired accuracy. This is achieved through slicing the biperiodic region into layers which are constant in z, so that the above solution can be utilized; the thinner the layers the higher the accuracy. The overall chain matrix is again calculated as the product of the chain matrices of the respective layers.

4 Numerical Example

In Figs. 2 and 3, the magnitude of the x-component of the total electric field is illustrated for the case, that an incident Gaussian beam at 65GHz (beam waists $w_{ox} = w_{oy} =$ 10mm) is splitted by a binary grating as shown in Fig. 1. The values of the design parameters are as follows: $\varepsilon_1 = \varepsilon_4 = 1$, $\varepsilon_2 = \varepsilon_3 = 2.25$, $d_1 = 8.7$mm, $d_2 = 5.8$mm, $d_3 = 15.0$mm, $d_4 = 30.5$mm, $L_x = 7.0$mm, $L_y = 8.3$mm, $b_x = 5.6$mm, $b_y = 3.3$mm. The incident beam has only an x-component, so that no y-component of the total electric field exists. The images have been produced by using a more elaborated version of the analysis presented here.

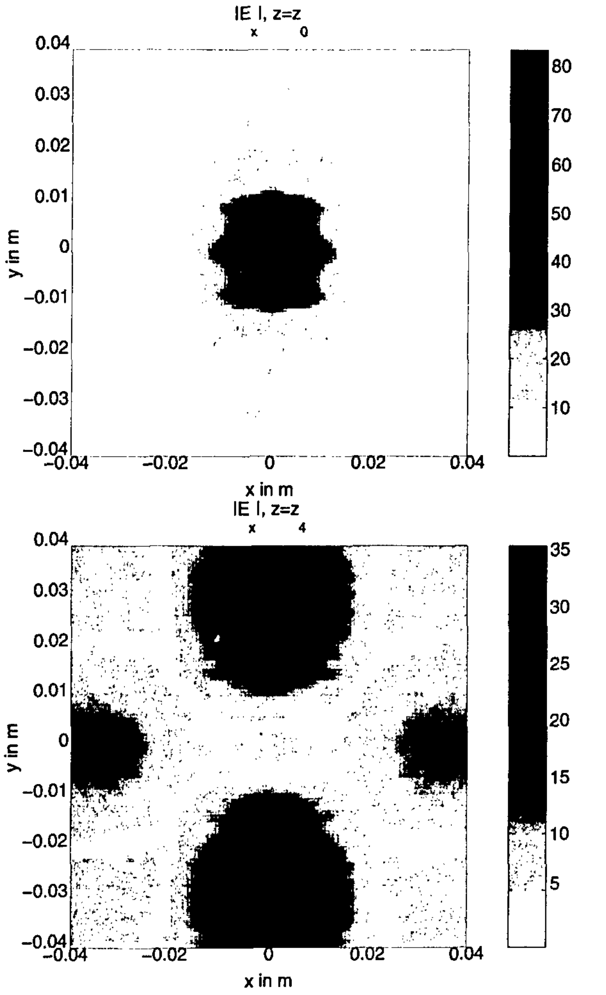

Figure 2: $|E_x|$-component of the total electric field in planes $z = z_0$ and $z = z_1$

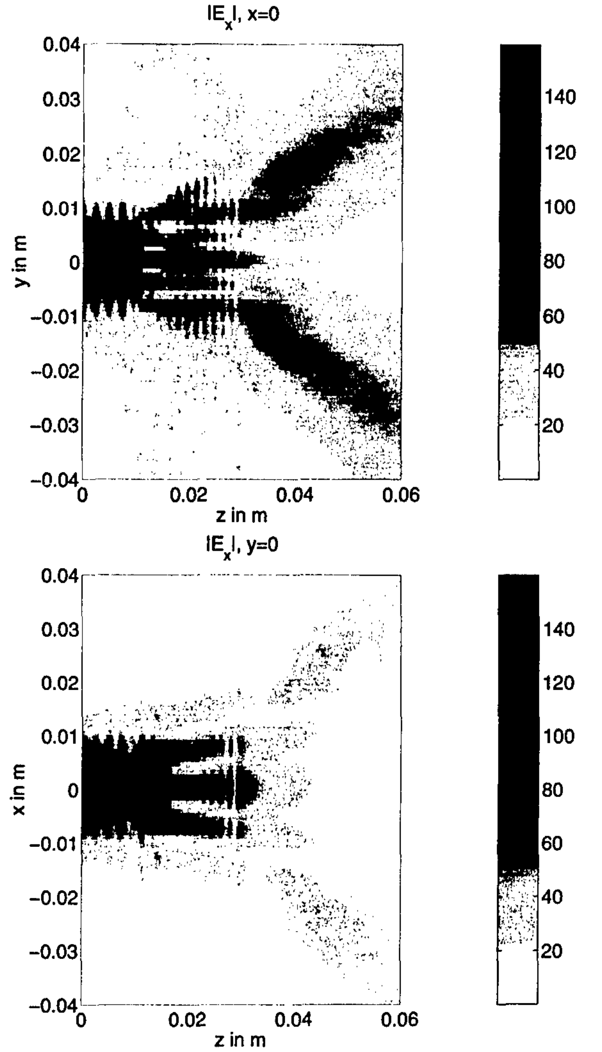

Figure 3: $|E_x|$-component of the total electric field in planes $x = 0$ and $y = 0$

5 Conclusions

Since all techniques applied in transmission line theory can also be applied to the diffraction of electromagnetic fields at biperiodic structures, the analysis presented still incorporates many imaginable extensions of the theory. For instance, reflection and transmission coefficients in matrix form can be defined in a similar fashion as has been shown for the propagation constant and the characteristic admittance/impedance. By introducing a complex permittivity, the analysis presented here can also be extended to metallic gratings. Furthermore, the diffraction of electromagnetic beams can be treated, if they are periodically shifted [7].

To summarize, an efficient and compact analysis method has been presented which allows the analytical solutions known from transmission line theory to be applied to electromagnetic fields diffracted by objects with periodic structure in one, two, and three dimensions. As indicated, further generalizations of the theory are possible.

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